

Notes on Ultrafilters and the Stone–Čech Construction

Alex Provost

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1 Introduction

Quick notes on ultrafilters and the Stone–Čech construction.

2 Filters and Ultrafilters

Definition. Let X be a set. A (proper) **filter** \mathcal{F} on X is a family $\mathcal{F} \subset \mathcal{P}(X)$ of subsets of X satisfying

- (1) $\emptyset \notin \mathcal{F}$.
- (2) $X \in \mathcal{F}$.
- (3) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$. (\mathcal{F} is closed under finite intersection.)
- (4) $B \supset A \in \mathcal{F} \implies B \in \mathcal{F}$. (\mathcal{F} is upward closed.)

Intuitively, a filter helps us locate elements of X by scanning its subsets. Note that filters satisfy the **finite intersection property (FIP)**: $A_1, \dots, A_n \in \mathcal{F} \implies \bigcap_{i=1}^n A_i \neq \emptyset$, and in particular, it can never happen that $A \in \mathcal{F}$ and also $A^c \in \mathcal{F}$.

Examples. (1) The **trivial filter** $\mathcal{F} = \{X\}$.

- (2) When X is infinite, the **cofinite** or **Fréchet filter** $\mathcal{F} = \{A \subset X \mid A^c \text{ is finite}\}$.
- (3) Any element $x \in X$ generates a **principal (ultra)filter** $\mathcal{F} = \langle x \rangle = \{x \in A \subset X\}$.
- (4) More generally, given any family $S \subset \mathcal{P}(X)$ satisfying the FIP, the **filter generated by S** is given by the upward closure of the finite intersections of S :

$$\mathcal{F} = \langle S \rangle = \{\bigcap_{i=1}^n A_i \subset A \subset X \mid A_i \in S\}.$$

Thus principal filters are equivalent to filters generated by singletons: $\langle x \rangle = \langle \{x\} \rangle$.

Definition. A filter \mathcal{F} is an **ultrafilter** if for any $A \subset X$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Hence an ultrafilter is a perfect element locator: either the element we seek is in a subset or its complement.

Examples. (1) Neither the trivial filter nor the cofinite filter are ultrafilters (the former is clear; for the latter, consider an infinite subset with an infinite complement).

- (2) A principal filter $\langle x \rangle$ is an ultrafilter since any subset either contains x or does not.

Ultrafilters are characterized by some *finite* union property analogous to the FIP that holds for general filters:

Proposition 2.1. \mathcal{F} is an ultrafilter if and only if $A_1, \dots, A_n \notin \mathcal{F} \implies \bigcup_{i=1}^n A_i \notin \mathcal{F}$.

Proof. If \mathcal{F} is an ultrafilter and $A_i \notin \mathcal{F}$, then $A_i^c \in \mathcal{F}$ by the defining property of an ultrafilter. Hence $(\bigcup_{i=1}^n A_i)^c = \bigcap_{i=1}^n A_i^c \in \mathcal{F}$ since filters are closed under finite intersection. But then $\bigcup_{i=1}^n A_i \notin \mathcal{F}$, again by the defining property of an ultrafilter.

Conversely, any $A \subset X$ satisfies $A \cup A^c = X \in \mathcal{F}$, whence by the contrapositive of the finite union property either A or A^c must belong to \mathcal{F} . Thus \mathcal{F} is an ultrafilter whenever that property is satisfied. \square

Proposition 2.2. Any ultrafilter that contains a finite set is principal.

Proof. If an ultrafilter \mathcal{F} contains $\{x_1, \dots, x_n\} = \bigcup_{i=1}^n \{x_i\}$ then it must contain one of the $\{x_i\}$, by the contrapositive of Proposition 2.1. Since filters are upward closed, this implies that $\langle x_i \rangle \subset \mathcal{F}$ as subsets of $\mathcal{P}(X)$. Conversely, as \mathcal{F} contains $\{x_i\}$, any element of \mathcal{F} must intersect $\{x_i\}$ lest the FIP be violated. Hence $\mathcal{F} \subset \langle x_i \rangle$, and so $\mathcal{F} = \langle x_i \rangle$ is principal. \square

Corollary 2.3 (Ultrafilters on finite sets). Any ultrafilter on a finite set X is principal, and there is a natural bijection between X and the set βX of ultrafilters on X .

Proof. By Proposition 2.2, any ultrafilter $\mathcal{F} \in \beta X$ has the form $\mathcal{F} = \langle x \rangle$ for some $x \in X$, and moreover $x_1 \neq x_2 \implies \langle x_1 \rangle \neq \langle x_2 \rangle$ by the FIP. Hence $X \cong \beta X$ as sets. \square

As we will soon see, the set βX of ultrafilters on X may be endowed with a topology that makes it into a universal compactification of X , viewing the latter as a discrete topological space. Corollary 2.3 is then simply a restatement of the fact that a discrete space is finite if and only if it is compact.

Let us now turn our attention to potentially infinite sets X . There, it is no longer true that all ultrafilters are principal, but sadly there is no constructive way to obtain such non-principal, or **free**, ultrafilters.¹

Theorem 2.4 (Extension of filters to ultrafilters). Assuming the axiom of choice, any filter \mathcal{F} on a set X may be extended to an ultrafilter $\bar{\mathcal{F}}$ (not uniquely).

Proof. The filters on X that are finer than \mathcal{F} , i.e., those that are supersets of \mathcal{F} in $\mathcal{P}(X)$, form a poset $\mathcal{P}_{\supseteq \mathcal{F}}$ under set inclusion induced by $\mathcal{P}(X)$; we will apply Zorn's lemma on this poset to obtain a maximal element and prove that this is an ultrafilter.

First, let $\{\mathcal{F}_i\}_{i \in I}$ be a chain, i.e., a collection of filters in $\mathcal{P}_{\supseteq \mathcal{F}}$ that are totally ordered by inclusion. Then the ascending union $\bigcup_{i \in I} \mathcal{F}_i$ is an upper bound of the chain in $\mathcal{P}_{\supseteq \mathcal{F}}$: it is finer than \mathcal{F} because all the \mathcal{F}_i are and it is a filter, as we check:

- (1) $\emptyset \notin \bigcup_{i \in I} \mathcal{F}_i$ since $\emptyset \notin \mathcal{F}_i$ for all i .
- (2) $X \in \bigcup_{i \in I} \mathcal{F}_i$ since $X \in \mathcal{F}_i$ for all i .

¹The more common and general definition is that a filter \mathcal{F} is free if $\bigcap \mathcal{F} = \emptyset$. There exist non-free and non-principal filters: the filter on \mathbb{N} generated by the complements of even numbers is such an example. However, an ultrafilter is free if and only if it is non-principal, so there is no harm in using our definition if ultrafilters are all we care about.

- (3) $A, B \in \bigcup_{i \in I} \mathcal{F}_i \implies A \cap B \in \bigcup_{i \in I} \mathcal{F}_i$ since A, B lie in some common \mathcal{F}_i , which is closed under finite intersection.
- (4) $B \supset A \in \bigcup_{i \in I} \mathcal{F}_i \implies B \in \bigcup_{i \in I} \mathcal{F}_i$ since A lies in some \mathcal{F}_i , which is upward closed.

Hence by Zorn's lemma, $\mathcal{P}_{\supset \mathcal{F}}$ contains a maximal filter $\overline{\mathcal{F}}$, i.e., one which is not a proper subset of any other element of $\mathcal{P}_{\supset \mathcal{F}}$.

Suppose by way of contradiction that $\overline{\mathcal{F}}$ is not an ultrafilter; then there exists $A \subset X$ such that $A \notin \overline{\mathcal{F}}$ and $A^c \notin \overline{\mathcal{F}}$. Now each set in $\overline{\mathcal{F}}$, being non-empty, must intersect either A or A^c . But an even stronger statement holds: *at least one of A or A^c intersects all the sets in $\overline{\mathcal{F}}$* . Indeed, if this were not true then some $B \in \overline{\mathcal{F}}$ would lie in A while some other $C \in \overline{\mathcal{F}}$ would lie in A^c ; but then $B \cap C = \emptyset$, contradicting the closure under finite intersection of the filter $\overline{\mathcal{F}}$. Without loss of generality, we may thus assume that all of the members of $\overline{\mathcal{F}}$ meet A . But then $\overline{\mathcal{F}} \cup \{A\}$ satisfies the FIP, and the filter it generates is strictly finer than $\overline{\mathcal{F}}$, contradicting maximality. Hence $\overline{\mathcal{F}}$ is an ultrafilter extending \mathcal{F} .

Finally, this extension is not unique in general: the trivial filter $\{X\}$, for instance, admits every ultrafilter as an extension. \square

Note that the above process is non-constructive, in the sense that it relies on the axiom of choice and does not provide us with an explicit example of a free (i.e., non-principal) ultrafilter.

Proposition 2.5. *A free ultrafilter \mathcal{F} on an infinite set X is always finer than the cofinite filter.*

Proof. This says that any non-principal ultrafilter \mathcal{F} contains all the cofinite sets in $\mathcal{P}(X)$, or, taking the contrapositive, that if there exists a cofinite set $A \notin \mathcal{F}$, then \mathcal{F} must be principal; but this is just a restatement of what we already know. For if \mathcal{F} does not contain the cofinite set A , then since it is an ultrafilter it must contain its complement A^c , which is finite. Applying Corollary 2.2, the conclusion follows. \square

Hence ultrafilters on an infinite set X come in two very different varieties:

- (1) Principal ultrafilters, which are generated by very small sets (singletons). There are as many principal ultrafilters as there are elements of X .
- (2) Free ultrafilters, which are generated by very large sets (cofinite sets). The cardinality of the set of all free ultrafilters is strictly greater than that of the set of principal ultrafilters. In fact, if $|X| = \kappa$, then one can show that the cardinality of the set βX of all ultrafilters on X is $|\beta X| = 2^{2^\kappa}$. Hence "most" ultrafilters are free, even though we cannot explicitly describe a single one.

3 The Stone–Čech Compactification

3.1 Categorical preliminaries

Any set X may be thought of as a topological space by endowing it with the discrete topology. More formally, let \mathfrak{Set} and \mathfrak{Top} denote the categories of sets and topological spaces, respectively. "Discretization" is a functor $D : \mathfrak{Set} \rightarrow \mathfrak{Top}$ left adjoint to the forgetful functor $F : \mathfrak{Top} \rightarrow \mathfrak{Set}$ that forgets about the topology of a topological space. This simply means whenever X is a set and Y is a space, there is a natural correspondence

$$\text{hom}_{\mathfrak{Top}}(DX, Y) \cong \text{hom}_{\mathfrak{Set}}(X, FY)$$

between *continuous* maps from DX to Y and functions from X to FY . This adjunction is really just a restatement of the fact that any map from a discrete space is automatically continuous, and it is equivalent to the following universal property: for any function $f : X \rightarrow Y$, there is a unique continuous map $\bar{f} : DX \rightarrow Y$ making the following diagram commute:

$$\begin{array}{ccc} DX & & \\ \uparrow & \searrow \exists! \bar{f} & \\ X & \xrightarrow{f} & Y \end{array}$$

(Here and henceforth I am abusing notation by not writing down forgetful functors all over the place. In the diagram should really appear arrows $f : X \rightarrow FY$ and $F\bar{f} : F(DX) \rightarrow FY$ that live in \mathfrak{Set} .)

The discretization DX is always Hausdorff, but it will never be compact unless X is finite. To remedy this, we will construct a space βX which is both compact and Hausdorff and obtained from X "universally". More precisely, if $\mathfrak{Top}_{\mathfrak{CH}}$ denotes the category of compact Hausdorff topological spaces, our goal is to obtain a functor $\beta : \mathfrak{Set} \rightarrow \mathfrak{Top}_{\mathfrak{CH}}$ which, in analogy to the discretization functor above, is left adjoint to the forgetful functor $F : \mathfrak{Top}_{\mathfrak{CH}} \rightarrow \mathfrak{Set}$. The corresponding universal property should look like this: for any function f from a set X to a compact Hausdorff space Y , there should exist a unique continuous map $\bar{f} : \beta X \rightarrow Y$ making the following diagram commute:

$$\begin{array}{ccc} \beta X & & \\ \uparrow & \searrow \exists! \bar{f} & \\ X & \xrightarrow{f} & Y \end{array}$$

Naturally, the construction of βX should also come with a suitable embedding $X \rightarrow \beta X$ (the vertical arrow in the diagram).

3.2 Construction of the Stone–Čech compactification βX

As we hinted at earlier, the Stone–Čech compactification βX of any set (or discrete topological space) X will consist of the set of all ultrafilters on X , endowed with a topology that makes it a universal compact Hausdorff space. Recall the following notion from point-set topology:

Definition. Let X be a set. A family $\mathcal{B} \subset \mathcal{P}(X)$ of subsets of X is called a **basis for a topology on X** if the following two conditions hold; in this case, elements of \mathcal{B} are called **basic open sets**.

- (1) $\bigcup \mathcal{B} = X$. (X is covered by basic open sets.)
- (2) For all $B_1, B_2 \in \mathcal{B}$, there exists some $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$. (Finite intersections of basic open sets always contains basic open sets.)

Then X may be endowed with a natural topology, the **topology generated by \mathcal{B}** , as follows: declare $U \subset X$ to be open whenever U may be expressed as a union of basic open sets. Thus

$$U \text{ is open in } X \iff \text{There exists a family } \{U_i\}_{i \in I} \text{ of basic open sets such that } U = \bigcup_{i \in I} U_i.$$

The two conditions above guarantee that this is a topology on X .

Definition (Topology of βX). If X is any set, let βX denote the set of all ultrafilters on X :

$$\beta X = \{\mathcal{F} \subset \mathcal{P}(X) \mid \mathcal{F} \text{ is an ultrafilter}\}.$$

The **basic open sets for the topology on βX** are those of the form

$$[A] = \{\mathcal{F} \in \beta X \mid A \in \mathcal{F}\},$$

where A ranges over all subsets of X . Thus the assignment $A \mapsto [A]$ is a mapping $\mathcal{P}(X) \rightarrow \mathcal{P}(\beta X)$.

Before proving that this defines a topology which makes βX into a compact and Hausdorff space, we will prove certain algebraic properties about $[A]$ which will come in handy in the subsequent manipulations.

Lemma 3.1 (Algebraic properties of $[\cdot]$). *Let $A, B \in \mathcal{P}(X)$. Then the map $[\cdot] : \mathcal{P}(X) \rightarrow \mathcal{P}(\beta X)$ defined above has the following properties:²*

- (a) $[\emptyset] = \emptyset$.
- (b) $[X] = \beta X$.
- (c) $[A \cup B] = [A] \cup [B]$.
- (d) $[A \cap B] = [A] \cap [B]$.
- (e) $A \subset B \iff [A] \subset [B]$.
- (f) $A = B \iff [A] = [B]$.
- (g) $[A^c] = [A]^c$.

Proof. (a) $[\emptyset] = \emptyset$ since no ultrafilters contain the empty set.

(b) $[X] = \beta X$ since all ultrafilters contain X .

(c) $[A \cup B] \subset [A] \cup [B]$ is a consequence of the "finite union property" for ultrafilters described in Proposition 2.1. The converse $[A \cup B] \supset [A] \cup [B]$ holds by the upward closure property of filters: if \mathcal{F} contains A or B then it must contain $A \cup B$.

(d) $[A \cap B] \subset [A] \cap [B]$ holds by the upward closure property of filters: if \mathcal{F} contains $A \cap B$ then it must contain A and B . The converse $[A \cap B] \supset [A] \cap [B]$ holds because filters are closed under finite intersection.

(e) $A \subset B \implies [A] \subset [B]$ follows from (c) since

$$A \subset B \iff A \cup B = B \implies [A] \cup [B] = [A \cup B] = [B] \iff [A] \subset [B].$$

Conversely, assume $[A] \subset [B]$, i.e. every ultrafilter containing A also contains B . Let $x \in A$ be arbitrary; then the principal ultrafilter $\mathcal{F} = \langle x \rangle$ contains A , so by hypothesis it also contains B . But this simply says that $x \in B$. Hence we have shown that any x in A is also in B , whence $A \subset B$.

(f) $A = B \iff [A] = [B]$ follows immediately from (e) and antisymmetry of the partial order relation \subset .

(g) $[A^c] = [A]^c$ follows from the first four properties: (b) and (c) imply that

$$[A] \cup [A^c] = [A \cup A^c] = [X] = \beta X$$

while (a) and (d) imply that

$$[A] \cap [A^c] = [A \cap A^c] = [\emptyset] = \emptyset.$$

□

²Concisely, $[\cdot]$ is an injective homomorphism of Boolean algebras.

Lemma 3.2. *The family $\mathcal{B} = \{[A] \subset \beta X \mid A \in \mathcal{P}(X)\}$ is a basis for a topology on βX .*

Proof. (1) Take $A = X$; then by of Lemma 3.1(b), $[X] = \beta X$ already covers the whole space.

(2) By Lemma 3.1(d), if $A, B \in \mathcal{P}(X)$ then the intersection $[A] \cap [B] = [A \cap B]$ of basic open sets is itself a basic open set. □

Theorem 3.3. *Endow βX with the topology generated by the basic open sets of the form $[A]$. Then:*

- (1) βX is compact.
- (2) βX is Hausdorff.
- (3) βX is totally disconnected.³
- (4) X embeds as a dense subspace of βX by mapping $x \in X$ to the principal ultrafilter $\langle x \rangle \in \beta X$.⁴

Proof. (1) To prove compactness, it suffices to show that any cover of βX with basic open sets admits a finite subcover. Assume by way of contradiction that this is not so; then there exists a family $\{[A_i]\}_{i \in I}$ of basic open sets such that

$$\beta X = \bigcup_{i \in I} [A_i] \quad \text{but} \quad \beta X \neq [A_{i_1}] \cup \cdots \cup [A_{i_n}]$$

for any finite collection of indices $i_1, \dots, i_n \in I$. By Lemma 3.1(b)(c)(f), this is equivalent to

$$[X] \neq [A_{i_1} \cup \cdots \cup A_{i_n}] \quad \text{or, dropping filters,} \quad X \neq A_{i_1} \cup \cdots \cup A_{i_n}.$$

Taking complements, this simply says that $\emptyset \neq A_{i_1}^c \cap \cdots \cap A_{i_n}^c$, i.e., the family $\{A_i^c\}_{i \in I}$ satisfies the FIP; hence it generates a filter $\mathcal{F} = \langle \{A_i^c\}_{i \in I} \rangle$. Let $\overline{\mathcal{F}}$ be an ultrafilter extending \mathcal{F} (Theorem 2.4). Then $\overline{\mathcal{F}} \in \beta X = \bigcup_{i \in I} [A_i]$, so there exists some $i \in I$ such that $\overline{\mathcal{F}} \in [A_i]$, i.e. $A_i \in \overline{\mathcal{F}}$. But by construction, $A_i^c \in \mathcal{F} \subset \overline{\mathcal{F}}$. Since a filter cannot contain both a set and its complement, this is the contradiction we seek.

(2) and (3) To prove that βX is Hausdorff we will separate any two distinct ultrafilters $\mathcal{F}, \mathcal{F}' \in \beta X$ by basic open sets. Since $\mathcal{F} \neq \mathcal{F}'$ by hypothesis, we may assume without loss of generality that there is some $A \in \mathcal{P}(X)$ which belongs to \mathcal{F} but not \mathcal{F}' . Since \mathcal{F}' is an ultrafilter, this implies that $A \in \mathcal{F}$ and $A^c \in \mathcal{F}'$, i.e., $\mathcal{F} \in [A]$ and $\mathcal{F}' \in [A^c] = [A]^c$. (This last equality is Lemma 3.1(g).) Thus the basic open sets $[A]$ and $[A]^c$ separate \mathcal{F} and \mathcal{F}' as desired.

In fact, this argument shows that βX is totally disconnected, because $[A]$ and $[A]^c$ provide a disconnection between \mathcal{F} and \mathcal{F}' , which are arbitrary points in the space βX . Thus any distinct pair of points in βX cannot lie in the same connected component.

(4) To prove that the set of all principal ultrafilters is dense in βX it suffices to show that any non-empty basic open set $[A] \subset \beta X$ meets a principal ultrafilter. By assumption $[A]$ is non-empty, thus so is A (Lemma 3.1(a)(f)); pick any $x \in A$. Then $\{\langle x \rangle\} = [\{x\}] \subset [A]$ by Proposition 2.2 and Lemma 3.1(e), i.e., $[A]$ contains the principal ultrafilter $\langle x \rangle$. This completes the proof. □

³A space is **totally disconnected** if its connected components are singletons. A space satisfying the first three properties of the Theorem is called a **Stone space**.

⁴A compact space in which X embeds densely is called **compactification of X** . Hence βX is a compactification of X , the **Stone-Čech compactification of X** .

3.3 Neighborhood filters, continuity and convergence

Before tackling the universal property of βX , let us see how to reformulate the basic topological concepts of continuity and convergence in terms of filters. Recall that a **neighborhood** of a point in a topological space is a superset of an open set that contains the point.

Definition (Neighborhood filter). Let X be a topological space and $x \in X$ a point. The **neighborhood filter at x** is given by

$$\mathcal{N}_x = \{N \subset X \mid N \text{ is a neighborhood of } x\}$$

It is clear that \mathcal{N}_x is upward closed, and it is closed under finite intersection because that property is satisfied by open sets in a topological space. Hence \mathcal{N}_x is a filter, and clearly $\mathcal{N}_x \subset \langle x \rangle$. Intuitively, the ultrafilter $\langle x \rangle$ contains perfect information about the point x , while the filter \mathcal{N}_x knows only about x what it can gather from its neighborhoods. But neighborhoods are all we need to do topology!

Definition (Filter convergence). We say that a filter \mathcal{F} on a topological space X **converges to** $x \in X$ (or that x **is a limit of \mathcal{F}**) if $\mathcal{N}_x \subset \mathcal{F}$, i.e., if it contains all the neighborhoods of x . If this is the case we will write $\mathcal{F} \rightarrow x$, or sometimes $\lim \mathcal{F} = x$ when the limit is unique.

Thus a filter converges to x if it lies somewhere between the neighborhood filter \mathcal{N}_x and the "maximally convergent" ultrafilter $\langle x \rangle$.

Examples. (1) Both the filter \mathcal{N}_x and the principal ultrafilter $\langle x \rangle$ converge to x .

(2) If X is discrete, then $\mathcal{N}_x = \langle x \rangle$ is the only filter that converges to x .

(3) If X has the trivial topology, then $\mathcal{N}_x = \{X\}$ is the trivial filter for any $x \in X$, whence any filter converges to every point in X .

Definition (Pushforward of (ultra)filters along maps). Let $f : X \rightarrow Y$ be a function between sets and let \mathcal{F} be a filter on X . Then

$$f_*\mathcal{F} = \{A \subset Y \mid f^{-1}(A) \in \mathcal{F}\}$$

defines a filter on Y called the **pushforward of \mathcal{F} along f** . If \mathcal{F} is an ultrafilter then so is $f_*\mathcal{F}$. (That the pushforward really is an (ultra)filter follows from the fact that preimages preserve set inclusion, intersection and complementation.)

Lemma 3.4. *Pushforwards preserve the partial order on filters: $\mathcal{F} \subset \mathcal{F}' \implies f_*\mathcal{F} \subset f_*\mathcal{F}'$.*

Proof. If $\mathcal{F} \subset \mathcal{F}'$ then

$$A \in f_*\mathcal{F} \iff f^{-1}(A) \in \mathcal{F} \implies f^{-1}(A) \in \mathcal{F}' \iff A \in f_*\mathcal{F}'.$$

□

Recall that a function $f : X \rightarrow Y$ between topological spaces is **continuous** at $x \in X$ if for any neighborhood N of $f(x)$, the preimage $f^{-1}(N)$ is a neighborhood of x . In terms of filters, $N \in \mathcal{N}_{f(x)} \implies f^{-1}(N) \in \mathcal{N}_x$, or equivalently $\mathcal{N}_{f(x)} \subset f_*\mathcal{N}_x$.

Proposition 3.5 (Continuity in terms of filter convergence). *Let X, Y be topological spaces and $x \in X$ a point. Then $f : X \rightarrow Y$ is continuous at x if and only if $\mathcal{F} \rightarrow x$ implies $f_*\mathcal{F} \rightarrow f(x)$, i.e., the pushforward along f preserves convergence.*

Proof. Assume that f is continuous at x and $\mathcal{F} \rightarrow x$. The former means that $\mathcal{N}_{f(x)} \subset f_*\mathcal{N}_x$ and the latter means that $\mathcal{N}_x \subset \mathcal{F}$. By Lemma 3.4, $f_*\mathcal{N}_x \subset f_*\mathcal{F}$, whence $\mathcal{N}_{f(x)} \subset f_*\mathcal{F}$; but this is just the statement that $f_*\mathcal{F} \rightarrow f(x)$.

Conversely, specializing the hypothesis $(\mathcal{F} \rightarrow x) \implies (f_*\mathcal{F} \rightarrow f(x))$ to $\mathcal{F} = \mathcal{N}_x$ we obtain $f_*\mathcal{N}_x \rightarrow f(x)$, which once recast into the form $\mathcal{N}_{f(x)} \subset f_*\mathcal{N}_x$ is just the definition of continuity. \square

3.4 Universal property of βX

First, here is a useful characterization of the category $\mathfrak{Top}_{\mathfrak{c}\mathfrak{H}}$ in terms of ultrafilters:

Theorem 3.6 (Characterization of compact and Hausdorff spaces via ultrafilter convergence). *Let X be a topological space. Then:*

- (1) *X is compact if and only if every ultrafilter \mathcal{F} on X converges to at least one point.*
- (2) *X is Hausdorff if and only if every ultrafilter \mathcal{F} on X converges to at most one point.*

Hence the subcategory $\mathfrak{Top}_{\mathfrak{c}\mathfrak{H}} \subset \mathfrak{Top}$ of compact Hausdorff spaces is characterized by the property that all ultrafilters there have a well-defined limit.

Proof. (1) Assume that X is compact and that there exists an ultrafilter \mathcal{F} on X which does not converge to any point. This means that for any $x \in X$, there exists a neighborhood $N_x \notin \mathcal{F}$. These provide us with open neighborhoods $x \in A_x \subset N_x$ that together cover X , and $A_x \notin \mathcal{F}$ by upward closure. By the compactness hypothesis, there exists a finite subcover

$$X = A_{x_1} \cup \cdots \cup A_{x_n}.$$

But $X \in \mathcal{F}$, whence some $A_{x_i} \in \mathcal{F}$ by Proposition 2.1. This contradicts the fact that $A_{x_i} \notin \mathcal{F}$. Conversely, suppose that X is not compact. Then there is an open cover $X = \bigcup_{i \in I} A_i$ with no finite subcover. By the same argument as that in the proof of Theorem 3.3(1), there exists an ultrafilter \mathcal{F} containing all the sets A_i^c . Since \mathcal{F} is an ultrafilter, this means that \mathcal{F} does not contain any of the neighborhoods A_i ; in particular, \mathcal{F} cannot converge to any $x \in X$, since each belong to some A_i .

- (2) Assume that X is Hausdorff but that there exists some ultrafilter \mathcal{F} that converges to two distinct points x, x' . Take disjoint open sets $x \in A, x' \in A'$. Then $A, A' \in \mathcal{F}$ but $A \cap A' = \emptyset$, a contradiction.

Conversely, suppose that X is not Hausdorff. Then there are points $x \neq x'$ such that every neighborhood of x intersects every neighborhood of x' . This means that $\mathcal{N}_x \cup \mathcal{N}_{x'}$ has the FIP, and may thus be extended to an ultrafilter \mathcal{F} . But \mathcal{F} converges to both x and x' by construction. \square

Lemma 3.7. *If a filter \mathcal{F} on a topological space X converges to a point $x \in X$, then $x \in \bar{A}$ for each $A \in \mathcal{F}$.*

Proof. If there is some $A \subset X$ such that $x \notin \overline{A}$, then since \overline{A}^c is open there exists some neighborhood of x that does not meet A . Since $\mathcal{F} \rightarrow x$, this neighborhood belongs to \mathcal{F} , so A cannot also belong to \mathcal{F} lest the FIP fail. \square

Lemma 3.8. *If V is an open neighborhood of a point x inside a compact Hausdorff space, then there exists a neighborhood N of x such that $\overline{N} \subset V$.*

Proof. V^c is compact since it is a closed subspace of a compact space. For each $y \in V^c$, take disjoint open neighborhoods N_y and U_y of x and y respectively. As the U_y together cover the compact subspace V^c , there exists a finite subcover $U_{y_1} \cup \cdots \cup U_{y_n} \supset V^c$. Let

$$C = U_{y_1}^c \cap \cdots \cap U_{y_n}^c = (U_{y_1} \cup \cdots \cup U_{y_n})^c \subset V;$$

$$N = N_{y_1} \cap \cdots \cap N_{y_n}.$$

Then N is a neighborhood of x and $N \subset C \subset V$. Moreover, C is closed, so $\overline{N} \subset C \subset V$. \square

We finally have all the tools required to prove that the Stone–Čech construction is what we claimed it to be:

Theorem 3.9. *The functor $\beta : \mathbf{Set} \rightarrow \mathfrak{Top}_{\mathcal{E}\mathfrak{S}}$ is left adjoint to the forgetful functor.*

Proof. Remember that we have to prove the following assertion: given any function $f : X \rightarrow Y$, where X is a set and Y is a compact Hausdorff topological space, there is a unique continuous map $\overline{f} : \beta X \rightarrow Y$ making the following diagram commute:

$$\begin{array}{ccc} \beta X & & \\ \uparrow & \searrow \exists! \overline{f} & \\ X & \xrightarrow{f} & Y \end{array}$$

In this diagram, the vertical arrow is the embedding of X as a dense subspace of βX that maps x to the principal ultrafilter $\langle x \rangle$ (Theorem 3.3). Define

$$\overline{f}(\mathcal{F}) = \lim f_* \mathcal{F}.$$

This definition makes sense by Theorem 3.6 since Y is compact and Hausdorff. Moreover, the diagram commutes, as $\lim f_* \langle x \rangle = f(x)$.

We check that \overline{f} is continuous. Let $\mathcal{F} \in \beta X$ be an ultrafilter, and let $V \subset Y$ be an open neighborhood of $\overline{f}(\mathcal{F}) = \lim f_* \mathcal{F}$. We must verify that some open neighborhood of \mathcal{F} in βX gets mapped inside V through \overline{f} . Since Y is compact Hausdorff, we may apply Lemma 3.8 to obtain a neighborhood N of $\overline{f}(\mathcal{F})$ such that $\overline{N} \subset V$. By definition of the limit, $N \in f_* \mathcal{F}$, i.e., $U = f^{-1}(N) \in \mathcal{F}$. By definition of the topology on βX , $[U]$ is an open neighborhood of \mathcal{F} in βX . For each filter $\mathcal{F}' \in [U]$, we have $U \in \mathcal{F}'$ and so $f(U) \in f_* \mathcal{F}'$ by upward-closure (as $f^{-1}(f(U)) \supset U$). Hence by Lemma 3.7, $\overline{f}(\mathcal{F}') \in \overline{f(U)}$. Finally, since $f(U) = f(f^{-1}(N)) \subset N$, we have $\overline{f(U)} \subset \overline{N}$ and thus $\overline{f}(\mathcal{F}') \in \overline{N} \subset V$ as desired.

Finally, \overline{f} is unique, as two continuous functions that map into a Hausdorff space and agree on a dense subspace must coincide.⁵ \square

⁵Suppose that f, g agree on a dense subset but $f(x) \neq g(x)$. Take disjoint open sets $f(x) \in U$ and $g(x) \in V$; then $f^{-1}(U) \cap g^{-1}(V)$ is open and also non-empty because it contains x , thus it intersects the dense subset; but then U and V cannot possibly be disjoint.